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On the mechanism of blocking in a stratified fluid

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The steady, two-dimensional motion which can occur when a body moves horizontally at large Richardson number is examined. Theoretical evidence is presented for two propositions: (i) The nature of the motion depends on whether the vertical thickness of the body is large compared with an intrinsic length scale of the motion. (ii) If the body is sufficiently thick, then diffusion or heat conduction are important, even if the Schmidt or Prandtl number is large. The notion of 'near-similar' solutions (§ 4) is used to obtain a description of the motion past a thick body which is likely to approximate the real motion everywhere except fairly close to the body surface (§ 5). It predicts a very long wake, at the core of which is a blocking column, both fore and aft of the body (§ 5). The same prediction is implied for the two-dimensional Taylor column in a rotating system (§ 6).

1. Introduction

One of the striking effects of stable atmospheric stratification is the extent of 'upstream influence' of steady horizontal motion. It is due to the restraining effect of buoyant forces upon vertical motion of the ambient fluid. This is most pronounced when the moving body (solid or fluid) is wide in comparison with its height, so that the ambient fluid has to perform an effectively two-dimensional motion to let the body pass.

The first question about such upstream influence concerns the basic decay law of the motion very far from the body, and this was answered by Long. But he gave two answers, based on the respective assumptions that heat conduction and mass diffusion are negligible (Long 1959) or that either of these effects is important (Long 1962). Subsequent authors have adopted the first answer because the Schmidt number 'comparing' viscosity to diffusion is very large in the experimental realization in salt solutions (Yih 1959). But in the atmosphere, heat conduction and viscosity are generally of equal importance, and even in salt solutions some experimental observations (Yih 1959) are at variance with the predictions of the non-diffusive theory (Graebel 1969; Janowitz 1971). Indeed the size of non-dimensional parameters is an unreliable guide in circumstances affording ample distances over which small effects can accumulate.

The object of the following is to present evidence in support of the proposition

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that the physical mechanism of the upstream (and downstream) influence (or wake or blocking) depends on whether the moving body is thick or thin. This requires reference to the physical complexity of the fluid mechanism, which involves four independent length scales – against only one for the incompressible Navier–Stokes equations of homogeneous fluid – even without regard to any body dimension. The governing equations indicate a key role for the geometric mean l of these four scales, which is (table 2) about $\frac{1}{6}$ mm in salt solutions and about 2 cm in air. A body will be called thin in the following if its vertical thickness $2b$ is at most of the same order as the intrinsic scale l, and thick if

$$b/l = \theta \gg 1.$$

To decide between Long's two decay laws requires experimental evidence or theoretical analysis linking the far wake to the flow closer to the body. Pao's (1968) experiment has confirmed Long's (1959) non-diffusive decay law for the upstream wake of a horizontal plate pulled edgewise; it was thin, and the boundary layer (Martin & Long 1968) had zero displacement thickness. But for a vertical plate pulled broadside, Graebel's (1969) and Janowitz's (1971) proposed approximations predict a long upstream wake, but no downstream wake, whereas Yih's (1959) experiment shows both wakes and, in fact, indicates *fore-and-aft symmetry*. Such symmetry is unlikely to be compatible with the neglect of diffusion and heat conduction (Graebel 1969; appendix C). Furthermore, Graebel's proposal predicts a drag independent of fluid viscosity and even of body speed. Janowitz's (1971) proposal predicts an upstream wake with divergent momentum integral (appendix C), whence an infinite drag per unit span of the body. It cannot, therefore, approximate the description of a real flow either.

By contrast, we shall use the diffusive model (§ 2) to develop a description of steady, two-dimensional wakes of a vertical plate which is free from these defects. This description covers not only the far field, but extends closer to the body, and thus establishes an unequivocal connexion between the far field and the flow nearer the body. In particular, the drag is predicted to be

$$\pi\mu U b^2/l^2 = \pi\rho_0 U^2 b(\sigma Ri)^{\frac{1}{2}}.$$

The description near the body is qualitatively plausible, but stylized, rather than quantitatively realistic. However, the condition of zero normal velocity on the body surface is satisfied, and a plausible pressure distribution is there predicted.

The velocity field has fore-and-aft symmetry and the body pushes ahead of it a blocking column of fluid which is at rest relative to the body. Near the body, this column is nearly as thick as the body and is separated from the undisturbed atmosphere, above and below, by layers of strong shear and vertical density variation. With increasing distance from the body, these layers spread very gradually until they merge finally to transform the column into a jet spreading and weakening slowly with further distance. The approach to the ultimate similarity form of Long (1962) is surprisingly slow. In the lateral direction (up and down), the decay of the motion is more rapid and is oscillatory (in fact, the mass flow in the blocking column is balanced by backflow in the layers bounding that column, so that the total mass flow rate is zero through any plane across the wake and fixed with respect to the atmosphere). Such a structure of the velocity field appears in essential agreement with that observed by Yih (1959).

It is likely to be an approximate description of the real motion past a thick body, and particularly past a vertical flat plate, in the natural limit where the plate height $2b$ is large compared with the intrinsic scale l and small compared with the stratification scale $\rho_\infty(\partial\rho_\infty/\partial z)^{-1} = h$. A necessary qualification is that diffusivity (or heat conductivity), ϑ, be not too small; more precisely, the Richardson number $b^2 g/(U^2 h)$, where U is the body speed, must be large compared with the Schmidt (or Prandtl) number $\sigma = \nu/\vartheta$. Another way to express all this, in terms of the body Reynolds number $Ub/\nu = Re$ and the Boussinesq number $h/b = \beta$ is that our description (§ 5) is likely to approximate the exact description in the limit

$$\sigma Ri \to \infty, \quad Ri/\sigma \to \infty, \quad \beta \to \infty, \quad Re^2\sigma Ri \to \infty.$$

It will be observed that this is not a small Reynolds number limit, contrary to the assumptions made by most other authors on the subject. In fact, the usual approaches and arguments, so well established in other, rather similar contexts, turn out to be quite misleading here. Accordingly, we refrain pointedly in the following from approximating the exact nonlinear governing equations (§ 2), but rather proceed on a frank physical hunch and *test afterwards* the description to which it has led us. (This procedure also shows the results of Long (1959, 1962), Graebel (1968) and Janowitz (1971) to be approximations of a different type than had been rather generally accepted; see (31) below.)†

The analogy between two-dimensional motions in rotating and stratified fluids (Veronis 1970) permits us to draw corresponding conclusions for the structure and drag of Taylor columns in rotating systems at small Rossby and Ekman numbers (§ 6). There is no analogy between fluid motion in a rotating system and non-diffusive viscous motion in an atmosphere. The assumption of two-dimensional motion is restrictive in both atmospheres and rotating systems; it aims at the clear illumination of a basic mechanism.

2. Formulation

To obtain basic information on the far field of a body in steady horizontal motion, it is sufficient to consider a standard stratification

$$\rho_\infty = \rho_0(1 - z^*/h), \tag{1}$$

where z^* measures vertical height from the centre of the body and ρ_0, h are positive constants; h is usually very large compared with the body thickness, so

† The analysis is based on the assumption of unbounded extent of the fluid – axially in a rotating system, horizontally in a stratified atmosphere – but this appears to be less critical than had often been feared. The observations of the paper following show the *principal qualitative* features of the wakes in stratified fluid not to be destroyed by end walls at a reasonable distance from the moving body. In particular, the fore-and-aft symmetry is confirmed by these observations at parameter values consistent with (30) below (but see 'Note in review' on page 725).

that the artificial introduction of negative density at distant levels is of no practical concern.

We ask what steady, two-dimensional, incompressible flow can develop as the body is pulled in the horizontal direction of increasing x^*. In our frame fixed with respect to the body, then,

$$u^* \to -U, \quad w^* \to 0, \quad \rho^* \to \rho_\infty \quad \text{as} \quad |x^*| \to \infty. \tag{2}$$

Conservation of mass and momentum are expressed by

$$\operatorname{div}(\rho^* \mathbf{v}^*) = 0, \tag{3}$$

$$(\rho^* \mathbf{v}^* \cdot \operatorname{grad}) \mathbf{v}^* = -\operatorname{grad} p^* + \mu \nabla^2 \mathbf{v}^* + \rho^* \mathbf{g}, \tag{4}$$

and the hydrostatic pressure relative to that at $z^* = 0$ is thus

$$p_{st} = \rho_0 g[-z^* + z^{*2}/(2h)].$$

The stratification arises usually from temperature or salinity variation. If heat conduction or salt diffusion is significant, then diffusion theory (Prandtl 1952; Bird, Stewart & Lightfoot 1960) shows conservation of energy or salt to be expressed to the first approximation by

$$(\mathbf{v}^* \cdot \operatorname{grad}) \rho^* = \vartheta \nabla^2 \rho^*, \tag{5}$$

with thermal or mass diffusivity ϑ which is constant to this approximation, as is the viscosity $\mu = \rho_0 \nu$.

$$\sigma = \nu/\vartheta$$

is called the Prandtl number, in the case of heat conduction, and the Schmidt number, in that of mass diffusion.

Equations (1) – (5) have already introduced four length scales, h, ν/U, U^2/g and ϑ/U. Their geometric mean,

$$l = (g\vartheta\nu/g)^{\frac{1}{4}}$$

is a property of the fluid independent of any body or motion. We therefore transform to non-dimensional perturbation variables by

$$\left.\begin{aligned} x^* = lx, \quad z^* = lz, \\ u^*(x^*, z^*) = -U + Uu(x,z), \quad w^*(x^*,z^*) = Uw(x,z), \\ \rho^*(x^*,z^*) = \rho_0 \tilde{\rho} = \rho_\infty + \epsilon^{-3}\lambda\rho_0\rho(x,z), \\ p^*(x^*,z^*) = p_{st} + (\mu U/l)\, p(x,z), \end{aligned}\right\} \tag{6}$$

where $\lambda = l/h$ and $\epsilon^3 = \sigma^{-\frac{1}{4}}(\nu/Uh)^{\frac{1}{2}}(gh/U^2)^{\frac{1}{4}}$. Thus

$$\tilde{\rho} = 1 - \lambda z + \epsilon^{-3}\lambda\rho,$$

and Uu, Uw are also the absolute velocity components with respect to the undisturbed atmosphere. Some typical values of these scales and parameters are shown in tables 1, 2 and 3.

With the abbreviations

$$\alpha = \frac{Ul}{\nu}, \quad D = \tilde{\rho}(u-1)\frac{\partial}{\partial x} + \tilde{\rho} w \frac{\partial}{\partial z}, \quad \nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial z^2},$$

	b	$\nu\,(\mathrm{cm^2\,s^{-1}})$	$\vartheta\,(\mathrm{cm^2\,s^{-1}})$	h (cm)	$g\,(\mathrm{cm\,s^{-2}})$
Yih (1959) (salt water)	1 cm	0·013	$1{\cdot}1\times10^{-5}$	500	980
Air		0·14	0·19	$9{\cdot}4\times10^{5}$	980

TABLE 1

	Yih (1959)		Air	
l (cm)	0·0166		2·26	
λ	$3{\cdot}3\times10^{-5}$		$2{\cdot}4\times10^{-6}$	
$U\,(\mathrm{cm\,s^{-1}})$	0·02	0·2	5	1200
ϵ	0·323	0·150	0·256	0·0407
α	0·025	0·252	80·3	$2{\cdot}02\times10^{4}$
α/ϵ	0·078	1·69	330	5×10^{5}

TABLE 2

	Yih (1959)		Air			
b	1 cm		20 m		100 m	
$\theta\ (\equiv b/l)$	61		890		4450	
$\lambda\theta$	2×10^{-3}		2×10^{-3}		10^{-2}	
$U\,(\mathrm{cm\,s^{-1}})$	0·02	0·2	5	1200	5	1200
α/θ	4×10^{-4}	4×10^{-3}	0·09	23	0·02	4·5
$\epsilon^{-3}\theta^{-1}$	0·48	4·8	0·07	17	0·013	3·3
$\epsilon\theta$	20	9	230	36	1100	180
$\alpha\epsilon^{-3}\theta^{-2}$	2×10^{-4}	2×10^{-2}	6×10^{-3}	380	$2{\cdot}5\times10^{-5}$	15
$\epsilon^{6}\lambda\theta^{3}$	10^{-2}	10^{-4}	0·5	10^{-5}	60	10^{-3}

TABLE 3

(3)–(5) take the forms

$$\bar{\rho}^2\left(\frac{\partial u}{\partial x}+\frac{\partial w}{\partial z}\right) = \lambda\bar{\rho}w-\epsilon^{-3}\lambda D\rho, \tag{7}$$

$$\alpha Du = -\partial p/\partial x+\nabla^2 u, \tag{8}$$

$$\alpha Dw = -\partial p/\partial z+\nabla^2 w-\rho, \tag{9}$$

$$\epsilon^{-3}\bar{\rho}^{-1}D\rho-w = \nabla^2\rho. \tag{10}$$

The standard procedure is now to look for an approximation to a solution of (7)–(10) as a parameter tends to zero (e.g. the Reynolds number (Graebel 1969; Janowitz 1971)) by neglecting the terms it multiplies in (7)–(10). This turns out to be misleading in the problem at hand. A somewhat better heuristic principle is needed, and we propose the following. If T is an operator and we ask whether a function g can approximate a solution f of $Tf = 0$, we can look at Tg in a limit of interest. It is then relatively straightforward to split Tg into dominant terms and terms that tend to zero by comparison. If the dominant terms cancel by virtue of the choice of g, then g will be called a '*limit solution*' in our limit, because $Tg \to 0$ in a meaningful way.

As an example, the Falkner–Skan similarity solutions of classical boundary-layer theory are exact solutions of the boundary-layer equations, but those are only approximations to the (incompressible) Navier–Stokes equations in a certain limit B. As exact solutions, they lack relevance because they satisfy only special initial conditions, and the slightest change in those conditions generally destroys the special symmetry necessary for a solution of similarity form. In short, they are only isolated, special solutions of approximate equations, while our interest is in approximate solutions to the Navier–Stokes equations. The real relevance of the similarity solutions derives from the fact that they are representatives of classes of boundary-layer solutions within each of which all the solutions have a common asymptotic behaviour (Serrin 1967) in a certain limit – which is just limit B. They are, in fact, limit solutions of the (incompressible) Navier–Stokes equations in that limit B.

That g is a limit solution of $Tf = 0$ does not establish that g is an asymptotic approximation to a solution f; it only makes g a candidate for such an approximation. But g cannot even be such a candidate in a given limit unless it is a limit solution in that limit. Of course, g cannot usually be expected to be a limit solution uniformly throughout the space of the independent variables.

Further, the distinctions between parameter limits and coordinate limits (Chang 1961) and mixtures of them must be borne in mind. Thus the ultimate decay law of the upstream wake in our problem must be a limit solution in the limit $x \rightarrow +\infty$. A limit such as A: $x \rightarrow +\infty$ with $\lambda x^{\frac{1}{2}} \rightarrow 0$ will be called a conditional limit because an experiment must usually proceed with fixed parameters λ, etc., and however small λ, limit A must then fail for all sufficiently large x. A limit solution in such a conditional limit A can only be an "asymptotic transient" (Meyer 1967) with respect to the true co-ordinate limit $x \rightarrow +\infty$. None the less, such a transient may be all that is readily observable.

Our search for limit solutions of (7)–(10) will be assisted by the use of a simpler, associated set of equations as heuristic guide or 'comparison' equations, and it will be unnecessary to inquire how those are related to (7)–(10).† We choose

$$\left.\begin{aligned} \partial u/\partial x + \partial w/\partial z = 0, \quad \partial p/\partial x = \partial^2 u/\partial z^2,\\ \partial p/\partial z = -\rho, \quad -w = \partial^2 \rho/\partial z^2, \end{aligned}\right\} \tag{11}$$

and propose to study relevant solutions of these with a view to testing whether they are limit solutions of the exact equations (7)–(10).

† *Note in review.* The governing equations (7)–(10) contain no Reynolds number. While α may look superficially like a Reynolds number, no body dimension has been defined yet and, moreover, the length scale l in (6) is itself dependent on viscosity, so that α is not proportional to the reciprocal of the viscosity. None the less, we expect the inertia terms in (7)–(10) to be relatively unimportant in our problem because it is characteristic of steady wakes that the diffusion of momentum by shear is the primary process in them. Inertia enters in a secondary manner because the situation does not fit the very restrictive conditions of symmetry under which pure shear without fluid acceleration is possible in fluids. That shear is a primary process and fluid acceleration, only a secondary one, applies with particular force to the very long horizontal wakes expected in stable atmospheres. Indeed, their inviscid models (Yih 1965) are blocking columns of indefinite length. Another heuristic motivation for (11) has been given by Childress and Carrier (see Moore & Saffman 1969).

Experiment (Yih 1959) indicates the horizontal wakes of a vertical plate† to be symmetrical both with respect to the horizontal plane $z = 0$ (because there is no lift) and to the plane of the plate, i.e. $u(x,z)$ is even in both x and z, and $w(x,z)$ is odd in both. Accordingly, we consider only motions with that symmetry in what follows. For solutions of (11), this implies a pressure perturbation p even in z and odd in x, and a density perturbation ρ odd in both, and it will suffice henceforth to consider $x > 0$. The exact equations (7)–(10) admit solutions with almost this symmetry of u and w, if the Boussinesq parameter λ is sufficiently small; p and ρ then have the noted symmetry in z, but not in x.

3. Similarity

The search for limit solutions for the far field is simplified by the observation that the ultimate far field cannot depend on details of body shape and hence must be invariant under affine transformations. Long (1962) has therefore proposed

$$\left.\begin{aligned} p_s = (3x)^{-\frac{1}{3}} h(\eta) \qquad \rho_s &= (3x)^{-\frac{2}{3}} r(\eta), \\ u_s = (3x)^{-\frac{2}{3}} f'(\eta), \quad w_s &= (3x)^{-\frac{4}{3}} g(\eta), \\ \eta = (3x)^{-\frac{1}{3}} z&, \end{aligned}\right\} \tag{12}$$

with

$$\left.\begin{aligned} g' = 2f' + \eta f'', \quad f''' &= -h - \eta h', \\ h' = -r, \qquad\qquad r'' &= -g \end{aligned}\right\} \tag{13}$$

for the far upstream wake. (It satisfies (11).) Substitution into (7), multiplied throughout by $(3x)^{\frac{5}{3}}$, gives

$$\begin{aligned} (\eta f'' + 2f' - g')\,[1 - \lambda(3x)^{\frac{1}{3}}\eta - \epsilon^{-3}\lambda(3x)^{-\frac{2}{3}} r] = {} & -\lambda(3x)^{\frac{1}{3}} g - \epsilon^{-3}\lambda(2r + \eta r') \\ & + 2\lambda(3x)^{-\frac{2}{3}} f' r + \epsilon^{-3}\lambda(3x)^{-\frac{2}{3}} r'(\eta f' - g). \end{aligned}$$

Thus (12) is a limit solution of (7) in the conditional limit $x \to \infty$ for fixed η with $\lambda x^{\frac{1}{3}} \to 0$ and $\epsilon^{-3}\lambda \to 0$, if $f, g, r \in C^1(-\infty, \infty)$ and the first of (13) is satisfied. Substitution in (8)–(10) similarly leads to the other equations of (13) and further limits, all of which are summarized by

$$x \to \infty, \quad \eta \text{ fixed}, \quad \alpha x^{-\frac{1}{3}} \to 0, \quad \lambda x^{\frac{1}{3}} \to 0, \quad \epsilon^{-3} x^{-\frac{1}{3}} \to 0. \tag{14}$$

Thus x must be large, and the Boussinesq parameter λ small, but ϵ not too small, for (12) to be a candidate for a (transient) asymptotic approximation to a fluid motion.

For u even, and w odd, in z, (13) integrates to $f'' + \eta h = 0$, $h'' - \eta f = h''(0)$, and so if

$$\tau = j^{-1}\eta, \quad \xi = j\eta, \quad j = \exp\left(\tfrac{1}{6} i\pi\right)$$

† *Note in review.* The authors are indebted to Dr F. K. Browand and Dr C. D. Winant for communication of an account (to be published in *Geophysical Fluid Dynamics*) of experiments in a parameter range different from (30) below, which show a motion *without* fore-and-aft symmetry. Further work appears necessary to tell whether these lend more support to a diffusive or non-diffusive mechanism in their parameter range.

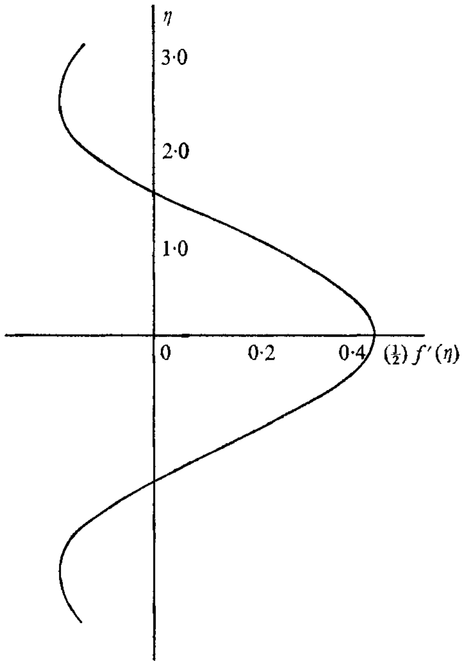

FIGURE 1. Similarity profile for the horizontal velocity of the far wake.

and $h+if = \phi(\tau)$, then

$$d^2f/d\xi^2 = \xi(f+i\phi), \quad d^2\phi/d\tau^2 = \tau\phi + j^2h''(0).$$

The unique solution giving bounded u_s as $|z| \to \infty$ for fixed x is

$$f(\eta) = -\tfrac{1}{2}\pi h''(0)\,[ij\,Ai(\xi) - ij^{-1}Ai(\tau) + j\,Gi(\xi) + j^{-1}Gi(\tau)] \tag{15}$$

(figure 1), where Ai is the Airy function and

$$Gi(z) = Ai(z)\int_0^z Bi(y)\,dy + Bi(z)\int_z^\infty Ai(y)\,dy$$

is discussed by Scorer (1950). Since $Gi''(z) = -\pi + z\,Gi(z)$,

$$\begin{aligned} h(\eta) &= \tfrac{1}{2}i\pi\,h''(0)\,[ij\,Ai(\xi) + ij^{-1}Ai\,(\tau) + j\,Gi\,(\xi) - j^{-1}Gi\,(\tau)] \\ &= O(\eta^{-4}) \quad \text{as} \quad |\eta| \to \infty, \end{aligned} \tag{16}$$

which is integrable on $(-\infty, \infty)$; by contrast,

$$f(\eta) \sim -\eta^{-1}h''(0) + O(|\eta|^{-7}), \qquad f'(\eta) = O(\eta^{-2}) \quad \text{as} \quad |\eta| \to \infty.$$

This limit solution describes a system of jets in the frame of an observer at rest with respect to the undisturbed atmosphere: a central jet in the direction of the body's motion is flanked by pairs of jets of alternating direction and rapidly decreasing strength (Long 1962; figure 1). The whole pattern spreads and decays according to the law (12).

Since $\lambda \to 0$ in (14), the limit solution satisfies the Boussinesq approximation and thus has a stream function. This tends to zero (like $|z|^{-1}$) with vertical distance

$|z|$ from the axis of symmetry $z = 0$, and the far wake therefore has zero displacement thickness. But it has a non-zero momentum integral. In this respect it must be noted that the hydrostatic pressure p_{st} is even less integrable than in an homogeneous atmosphere and now even fails to be odd in z. It is even in x, however, and therefore makes no contribution to the overall momentum balance. The momentum integral of the upstream wake may thus be taken as

$$\int_{-\infty}^{\infty} (p^* - p_{st} + \rho^* u^{*2} - \mu\, \partial u^*/\partial x^*)\, dz^* \sim \mu U \int_{-\infty}^{\infty} p\, dz = \mu U \int_{-\infty}^{\infty} h(\eta)\, d\eta \quad \text{as} \quad x \to \infty \tag{17}$$

by (6) and (12), and this exists, by (16). In fact (appendix A)

$$\int_{-\infty}^{\infty} h(\eta)\, d\eta = -\pi h''(0). \tag{18}$$

It is as well to note, however, that (14) makes no case for the practical value of this limit solution in the atmosphere or in laboratory experiments with salt solutions. The distance needed to approximate (14), and in particular, $\epsilon^{-3} x^{-\frac{1}{3}} \to 0$, to a reasonable degree, is often huge (Table 2). A similar analysis of Long's (1959) non-diffusive far-wake similarity law shows that law to be a limit solution of the exact equations (7)–(10) for

$$\left.\begin{array}{c} \epsilon x \to \infty, \quad \lambda \epsilon^{-\frac{3}{4}} x^{\frac{1}{4}} \to 0, \quad \alpha \epsilon^{-\frac{3}{2}} x^{-\frac{1}{2}} \to 0, \\ \lambda \epsilon^{\frac{15}{4}} x^{\frac{3}{4}} \to 0 \quad \text{and} \quad \epsilon^{\frac{3}{4}} x^{-\frac{1}{4}} z \text{ fixed}, \end{array}\right\} \tag{19}$$

which tends to be approximated at not quite so large distances (table 2). But this comparison is misleading, because it will emerge below that the large distances are required, not for diffusion to become important, but merely for the influence of body shape to die away so that a similarity law can be approximated.

It will help presently to have a similarly explicit representation for Long's (1962) diffusive similarity law

$$\left.\begin{array}{ll} p = (3x)^{\frac{1}{3}} k(\eta), & \rho = r(\eta) \\ u = g'(\eta), & w = (3x)^{-\frac{2}{3}} s(\eta) \\ \multicolumn{2}{c}{\eta = (3x)^{-\frac{1}{3}} z,} \end{array}\right\} \tag{20}$$

$$r = -k', \quad s = -r'', \quad s' = \eta g'', \quad g''' = k - \eta k' \tag{21}$$

for a boundary layer. If

$$u = \int_{\eta}^{\infty} q(t)\, dt,$$

so that $q(\eta) = -g''$, then $q'' = \eta k''$, $k^{iv} = -\eta q$ and the boundary conditions for a boundary layer above a solid wall at $z = 0$ are

$$u(0) - 1 = q^{iv}(0) = 2k'''(0) = s(0) = 0.$$

The unique solution is

$$q(\eta) = 3j^{-1} Ai\,(\xi) + 3j\, Ai\,(\tau),$$

the skin friction coefficient is

$$(l/U)\,\partial u^*/\partial z^*|_{z=0} = -3^{\frac{1}{2}}x^{-\frac{1}{3}}/\Gamma(\tfrac{2}{3})$$

and the displacement thickness is also zero.

To decide between (12) and the non-diffusive far-wake law for a thick body requires establishment of a connexion between the far wake and the flow closer to the body. Since details of the flow very close to the body are not of direct interest in this connexion, it suffices to consider a standard body, and the most convenient one is a vertical flat plate. The artificial singularities thereby introduced at the sharp edges

$$z^* = \pm b, \quad z = \pm b/l = \pm\theta, \quad x = 0$$

are of no concern in the present context, and the qualification 'except in a neighbourhood of the plate edges' is understood in all that follows.

On the inviscid fluid model (Yih 1965) the two-dimensional steady flow caused by the broadside motion of the plate consists of fore and aft blocking columns – in which the fluid is at rest relative to the plate – separated from the undisturbed atmosphere above and below by plane vortex sheets $z = \pm\theta$, $-\infty < x, y < \infty$. In a real fluid one then expects shear layers in the place of the vortex sheets, and experiment (Yih 1959) confirms that. Sufficiently near each plate edge, the shear layer must be independent of that issuing from the opposite edge, i.e. for $x \ll \theta$ the shear-layer structure must be invariant under affine transformations and hence must have similarity form. The whole flow at $x \gg 1$ can then be expected to be described by the superposition and interaction of these shear layers. This argument is fundamental to the analysis of magnetohydrodynamic wakes and Taylor columns in rotating fluids and applies equally to the upstream wake in non-diffusive stratified fluid (Graebel 1969). It is important to understand that this well-established argument fails as soon as heat conduction or mass diffusion are significant, because no similarity shear layer can then exist!

Indeed, a limit solution describing such a spreading shear layer must satisfy

$$u \to 0 \quad \text{as} \quad z \to +\infty, \qquad u \to 1 \quad \text{as} \quad z \to -\infty,$$

say. For the limit (14), that implies again (20), whence (21), and the general solution is again a linear combination of Airy functions of ξ and τ; but there is no such combination which is bounded for $-\infty < \eta < \infty$.

4. Near-similarity

A relevant solution of (11) must satisfy

$$u \to u_s(x,z) \quad \text{as} \quad \delta = (3x)^{-\frac{1}{3}}\theta \to 0$$

because the wake must approach similarity form ultimately, and (12) is the only similarity solution of (11) that possesses a non-zero momentum integral. Now, from (12) and (15) it can be shown (as in appendix A) that

$$\begin{aligned} u_s(x,z) &= -h''(0)\,(3x)^{-\frac{2}{3}}\int_0^\infty \tau \exp(-\tfrac{1}{3}\tau^3)\cos(\eta\tau)\,d\tau \\ &= -h''(0)\int_0^\infty t\exp(-t^3x)\cos(zt)\,dt, \end{aligned} \tag{22}$$

with unknown constant $h''(0)$. This suggests consideration of the functions

$$\left.\begin{aligned} u_F(x,z) &= \theta\int_0^\infty F(\theta t)\exp(-t^3x)\cos(zt)\,dt,\\ w_F(x,z) &= \theta\int_0^\infty t^2F(\theta t)\exp(-t^3x)\sin(zt)\,dt,\\ \rho_F(x,z) &= \theta\int_0^\infty F(\theta t)\exp(-t^3x)\sin(zt)\,dt,\\ p_F(x,z) &= \theta\int_0^\infty t^{-1}F(\theta t)\exp(-t^3x)\cos(zt)\,dt,\end{aligned}\right\} \tag{23}$$

which solve (11) for any function $F(\theta t)$ assuring absolute convergence of the integrals for all z and all $x > 0$. Then

$$u_F(z,x) = \delta\int_0^\infty F(\delta\tau)\exp(-\tfrac{1}{3}\tau^3)\cos(\eta\tau)\,d\tau$$

with again $$\delta = (3x)^{-\frac{1}{3}}\theta,\quad \eta = (3x)^{-\frac{1}{3}}z,$$

and thus if $F(y)$ is a power of y, then u_F has similarity form; in particular, if $F(y) = y$, then $u_F = \text{constant}\times u_S$. Otherwise, (23) has no similarity form, but if $F(y)$ is smooth enough at $y = 0$, for instance, if $F(y) = \text{constant}\times y^n + o(y^n)$ for some integer $n \geqslant 0$ and if $F(y)$ is bounded, then (23) may be called a '*near-similar*' solution of (11) because, as $\delta \to 0$,

$$u_F \sim \text{constant}\times\delta^{n+1}\int_0^\infty \tau^n\exp(-\tfrac{1}{3}\tau^3)\cos(\eta\tau)\,d\tau,$$

i.e. (23) approaches similarity form far upstream.

A near-similar solution (23) thus has a non-zero momentum integral if, and only if, $\{u_F, w_F, \rho_F, p_F\} \to \{u_S, w_S, \rho_S, p_S\}$ as $\delta \to 0$, and that is assured if

$$\lim_{y\to 0} F(y)/y = c > 0 \tag{24}$$

(and, say, F is bounded so that the integrals converge well), for then

$$u_F \to c\delta^2\int_0^\infty \tau\exp(-\tfrac{1}{3}\tau^3)\cos(\eta\tau)\,d\tau = \text{constant}\times u_S,$$

and similarly for w, ρ and p.

The determination of F requires conditions reflecting the nature of the motion nearer to the plate, though (14) raises doubts whether the near-similarity approach can cover more than a far-field approximation. However, two compatibility conditions at $x = 0$ are plausible: First, the relative normal velocity on the plate should vanish, i.e. $u(0,z) = 1$ for $|z| < \theta$. Secondly, the 'motions' fore and aft should join smoothly, away from the plate, and since the proposed symmetry (§ 2) requires a hydrodynamic 'pressure' perturbation p odd in x, it follows that $p(0,z) = 0$ for $|z| > \theta$. Thus F must satisfy the dual integral equations

$$\left.\begin{aligned} \lim_{x\to+0}\theta\int_0^\infty F(\theta t)\exp(-t^3x)\cos(zt)\,dt &= 1 \quad\text{for}\quad |z| < \theta,\\ \lim_{x\to+0}\theta\int_0^\infty t^{-1}F(\theta t)\exp(-t^3x)\cos(zt)\,dt &= 0 \quad\text{for}\quad |z| > \theta.\end{aligned}\right\} \tag{25}$$

Now, since our interest is not in a solution of (11), but in a candidate for an approximate solution of (7)–(10), there is no reason at this point for avoiding purely formal manipulation. Then (25) can be replaced by

$$\left.\begin{aligned} u_F(+0,z) &= \theta\int_0^\infty F(\theta t)\cos(zt)\,dt = 1 \quad \text{for} \quad |z| < \theta, \\ p_F(+0,z) &= \theta\int_0^\infty t^{-1}F(\theta t)\cos(zt)\,dt = 0 \quad \text{for} \quad |z| > \theta, \end{aligned}\right\} \tag{26}$$

(cf. Moore & Saffman 1969, equations (8.11, 12); Graebel 1969, equations (18)). By (24), $F(y)$ can be defined as an odd function on $(-\infty,\infty)$, so that the even function $F(\theta t)/t$ has formal Fourier transform

$$T[F(\theta t)/t] = \int_{-\infty}^\infty t^{-1}F(\theta t)\,e^{izt}\,dt = 2p_F(+0,z)/\theta.$$

Similarly,

$$\begin{aligned} T[F(\theta t)\operatorname{sgn} t] &= \int_{-\infty}^\infty F(\theta t)\operatorname{sgn} t\times e^{izt}\,dt = 2u_F(+0,z)/\theta \\ &= \int_{-\infty}^\infty t\operatorname{sgn} t\times t^{-1}F(\theta t)\,e^{izt}\,dt \\ &= \frac{1}{2\pi}T[t\operatorname{sgn} t]*T[F(\theta t)/t]. \end{aligned}$$

Hence,

$$\begin{aligned} u_F(+0,z) &= \frac{1}{2\pi}T[t\operatorname{sgn} t]*p(+0,z) = \frac{1}{\pi}[z^{-1}*\delta'(z)]*p_F(+0,z) \\ &= \frac{1}{\pi}\int_{-\infty}^\infty p'(\xi)\,(z-\xi)^{-1}\,d\xi, \end{aligned}$$

with principal value integral and $p'(z) = \partial p_F(+0,z)/\partial z$, and by (26),

$$\pi u_F(+0,z) = \int_{-\theta}^{\theta} p'(\xi)\,(z-\xi)^{-1}\,d\xi = 1 \quad \text{for} \quad |z| < \theta.$$

This is a well-known singular integral equation of which the only solution that is integrable and Holder continuous on $(-\theta,\theta)$ and odd in z is

$$p'(z) = -z(\theta^2-z^2)^{-\frac{1}{2}} \quad \text{for} \quad |z| < \theta.$$

Thus if the 'pressure' perturbation is continuous at the plate edges, it must be

$$p_F(+0,z) = \begin{cases} (\theta^2-z^2)^{\frac{1}{2}} & \text{for} \quad |z| < \theta \\ 0 & \text{for} \quad |z| > \theta, \end{cases}$$

and from (23) and (26), $F(y)$ must satisfy the integral equation

$$\theta\int_0^\infty t^{-1}F(\theta t)\cos(zt)\,dt = \begin{cases} (\theta^2-z^2)^{\frac{1}{2}} & \text{for} \quad |z| < \theta \\ 0 & \text{for} \quad |z| > \theta. \end{cases}$$

But that is a Fourier transform, and (Bracewell 1965) $F(y) = J_1(y)$, the Bessel function of first order.

5. A candidate

The near-similar expression (23) of direct interest is therefore

$$\begin{Bmatrix} u \\ w \\ \rho \\ p \end{Bmatrix}(x,z) = \theta\int_0^\infty J_1(\theta t)\exp(-t^3x)\begin{Bmatrix} \cos(zt) \\ t^2\sin(zt) \\ \sin(zt) \\ t^{-1}\cos(zt) \end{Bmatrix} dt \tag{27}$$

for $x > 0$, $-\infty < z < \infty$ (except at the plate edges), and u is even in x, while p, ρ and w are odd in x. This is to be tested as a limit solution of (7)–(10), but in what limit? The answer requires some discussion of the 'motion' represented by (27).

First, since $J_1(\delta\tau) \sim \frac{1}{2}\delta\tau$ as $\delta \to 0$ for bounded τ, and by (27), it can be confirmed (as in appendix C) that $u(x,z) \sim -\frac{1}{2}\theta^2 u_s/h''(0) + o(\delta^2)$ as $x \to \infty$ so that $\delta \to 0$, with η fixed, and similarly, $\{w, \rho, p\} \sim -\frac{1}{2}\theta^2\{w_s, \rho_s, p_s\}/h''(0)$. As intended, therefore, (27) tends to Long's similarity limit solution (12) at sufficiently large horizontal distances from the plate. It then exhibits the structure noted in § 3 (figure 1). This distant decay law, moreover, is now related to the body, at least to the extent that the arbitrary constant in Long's law is determined as

$$h''(0) = -\tfrac{1}{2}\theta^2,$$

whence (17) and (18) give the drag (contributed in equal parts by the upstream and dowstream wakes) of this 'motion' as

$$\pi\mu U\theta^2 = \pi\mu U(b/l)^2 = \pi\rho_0 U^2 b(\sigma Ri)^{\frac{1}{2}},$$

where $\sigma = \nu/\vartheta$ is the Prandtl or Schmidt number, and $Ri = b^2g/(U^2h)$, the Richardson number based on the plate height and speed. It may also be observed that the stream function is $O(z^{-1})$ for fixed $x \neq 0$, by Riemann's Lemma, so that the total 'mass-flow' in the wakes (relative to the undisturbed atmosphere) is zero.

Secondly, near the plate, as $\delta = (3x)^{-\frac{1}{3}}$, $\theta \to \infty$, it is found (appendix B) that

$$\lim_{\delta\to\infty} u(x,z) = \begin{cases} 1 & \text{for} \quad |z| < \theta, \\ -\theta^2(z^2-\theta^2)^{-\frac{1}{2}}[|z| + (z^2-\theta^2)^{\frac{1}{2}}]^{-1} & \text{for} \quad |z| > \theta, \end{cases}$$

$$\lim_{\delta\to\infty} p(x,z) = \begin{cases} (\theta^2-z^2)^{\frac{1}{2}} & \text{for} \quad |z| < \theta, \\ 0 & \text{for} \quad |z| > \theta, \end{cases}$$

$$\lim_{\delta\to\infty} \rho(x,z) = \begin{cases} z/(\theta^2-z^2)^{\frac{1}{2}} & \text{for} \quad |z| < \theta, \\ 0 & \text{for} \quad |z| > \theta, \end{cases}$$

and, at least in the distribution sense,

$$\lim_{\delta\to\infty} w(x,z) = \begin{cases} -3\theta^2 z/(\theta^2-z^2)^{\frac{5}{2}} & \text{for} \quad |z| < \theta, \\ 0 & \text{for} \quad |z| > \theta. \end{cases}$$

Thus the condition of zero normal 'velocity' at the plate is satisfied, and there are 'blocking columns' fore-and-aft of the plate. Above and below them, the only

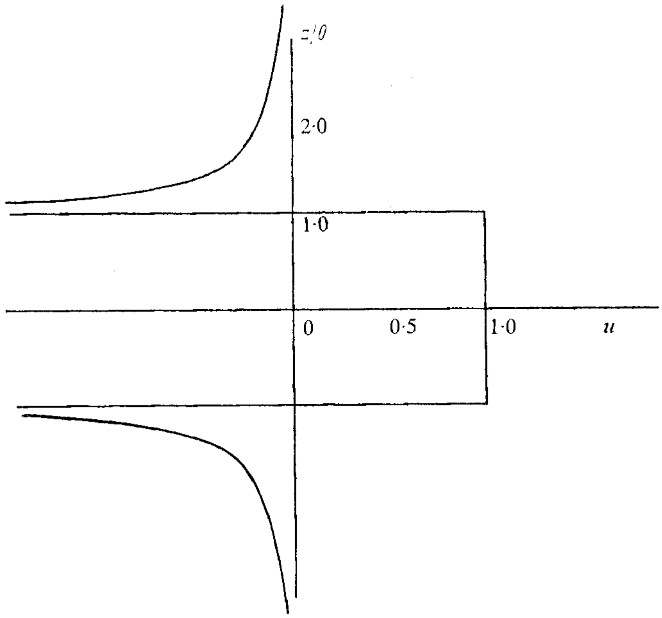

FIGURE 2. Horizontal velocity profile at $x = 0$, i.e. $\delta = \infty$.

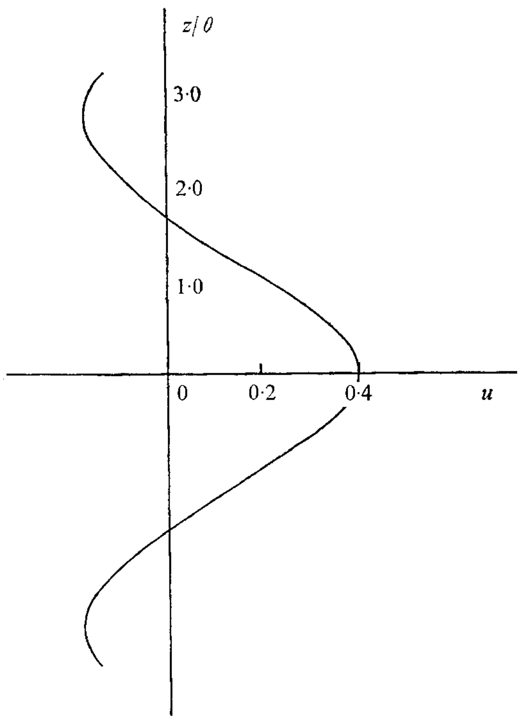

FIGURE 3. Horizontal velocity profile at distance from body corresponding to $\delta = (3x)^{-\frac{1}{3}}\,\theta = 1$.

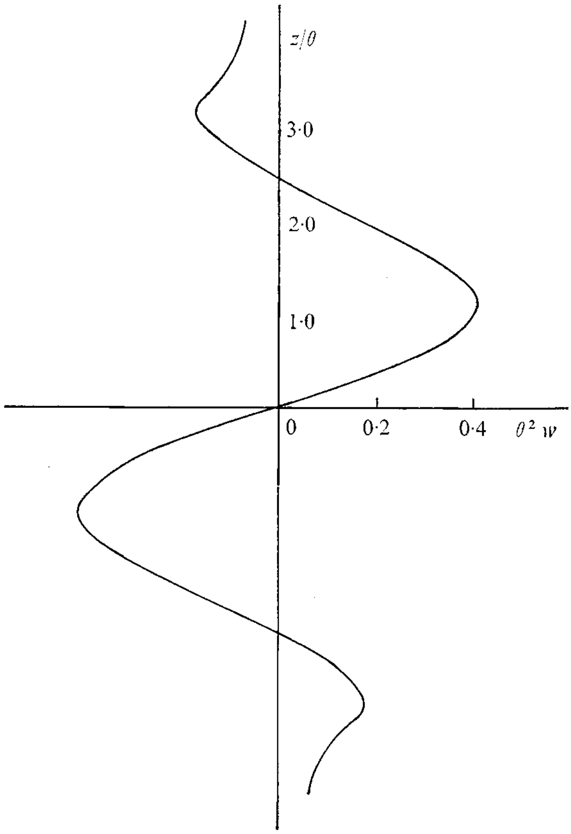

FIGURE 4. Vertical velocity profile at distance from body corresponding to $\delta = 1$. Note that the velocity scale is multiplied by θ^2.

perturbation is in the horizontal 'velocity', and this decays monotonely and algebraically with height (figure 2). The 'blocking columns' are bounded above and below by zones of high 'shear' adjacent to zones of high 'density' perturbation and rapid 'pressure' variation. Of course, this can at most be interpreted as a stylized representation of fluid motion, leaving out of account the secondary fluid acceleration implied by the shear and density variation. Despite the smoothing effect of diffusion, there is a marked 'density' deficiency at the lower edge of the 'blocking columns' and a corresponding excess, at the upper edge. Across the plate there is a 'pressure' jump which takes its maximum value of $2\theta\mu U/l = 2\rho_0 U^2(\sigma Ri)^{\frac{1}{2}}$ at the plate centre and exerts a force equal to the drag on the plate. If $\theta = b/l \gg 1$, the 'motion' even comes close to satisfying the no-slip condition at the plate, since $w = O(\theta^{-2})$ there, so that only a minor non-uniformity near the plate surface is then indicated (except at the edges). Of course, it is not intended here to study the flow very close to the plate, and the limit $\delta \to \infty$ only serves to indicate the structure of the 'motion' not too far from the plate.

To sketch the transition between the two limits, some representative profiles at distances corresponding respectively to $\delta = (3x)^{-\frac{1}{3}}\theta = 1$ and 3 are shown in figures 3–8. (For Yih's (1959) experiment with a plate 1 in. high, $\delta = 1$ at about

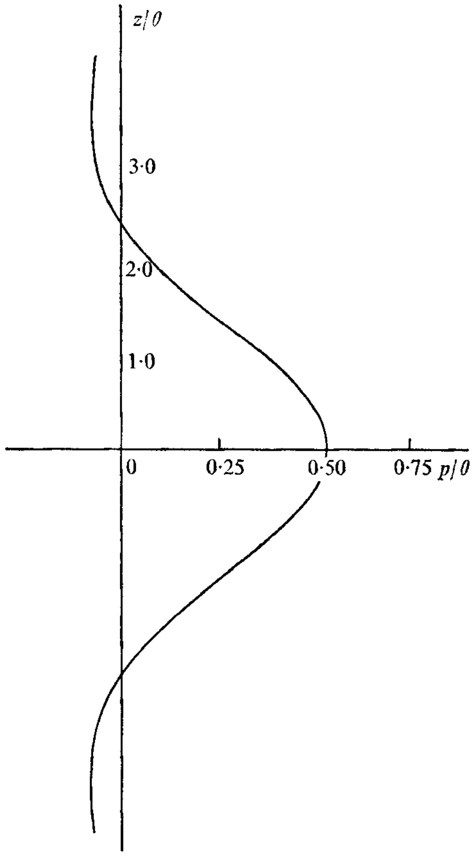

FIGURE 5. Pressure perturbation profile at distance from body corresponding to $\delta = 1$.

20 m from the plate, and $\delta = 3$ at about 60 cm distance from it.) Figure 7 shows values $u > 1$ near the axis of symmetry: the 'fluid' is not only blocked, but even accelerated backward, away from the plate. At these intermediate distances corresponding to $\delta = O(1)$, the 'shear' zones bounding the 'blocking column' spread and merge, and $\delta = 1$ thus characterizes the length of the 'blocking columns' fore-and-aft. In Yih's (1959) experiment they are thus about 20 m long!

The next concern is with the extent to which the inverted commas can be removed by showing the comparison 'motion' to be a limit solution of the governing equations (7)–(10). It is plausible that the near-similar expression (27) may be a limit solution under a less restrictive far-wake limit than (14). Actually, the structure of the motion in the middle field, and even in the region of the body corresponding to large δ, is fairly plausible and the 'motion' satisfies the physical boundary conditions on the plate to a surprising extent when it is thick ($\theta = b/l \gg 1$). The general decay and spread with increasing x implies, moreover, that the 'velocity', 'density' and 'pressure', and all their derivatives, are largest in the region of the body. In that region, the last set of limits shows θu, p/θ, ρ and $\theta^2 w$ to be essentially functions of z/θ and hence, a pure parameter limit based on $\theta \to \infty$, rather than on $x \to \infty$, is appropriate there.

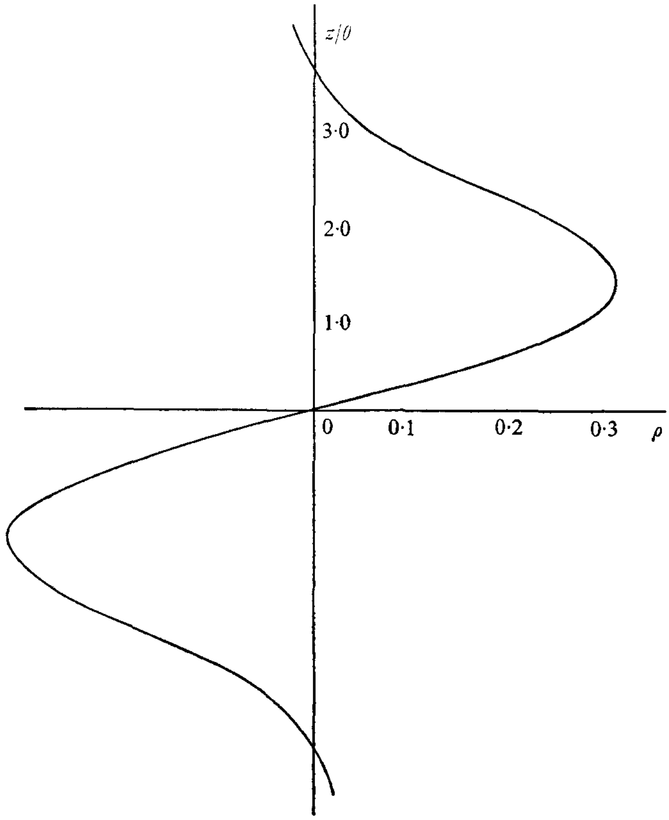

FIGURE 6. Density perturbation profile at distance from body corresponding to $\delta = 1$.

To test this, let $t = s/\theta$ in (27), so that

$$\begin{Bmatrix} u \\ \theta^2 w \\ \rho \\ p/\theta \end{Bmatrix} (x, z) = \int_0^\infty \begin{Bmatrix} A \\ s^2 B \\ B \\ s^{-1} A \end{Bmatrix} ds, \tag{28}$$

with

$$\begin{Bmatrix} A \\ B \end{Bmatrix} = J_1(s) \exp[-x(s/\theta)^3] \begin{Bmatrix} \cos \\ \sin \end{Bmatrix} (sz/\theta),$$

and substitute into (7)–(10). In the case of (10) for instance, this gives for $x > 0$

$$\epsilon^{-3}(\textstyle\int A\, ds - 1)(-\theta^{-1}\int s^3 B\, ds) + \epsilon^{-3}(\int s^2 B\, ds)\,\theta^{-1}\int sA\, ds - \int s^2 B\, ds$$
$$= \theta^{-4}\textstyle\int s^6 B\, ds - \int s^2 B\, ds,$$

and reference to (27) shows that these integrals remain bounded as $\theta \to \infty$ for fixed δ. Hence, (28) is seen to satisfy (10) in the limit $\theta \to \infty$, $\epsilon^{-3}\theta^{-1} \to 0$. Similar substitution in (7)–(9) shows (27) to be a limit solution of (7)–(10) in the parameter limit

$$\theta \to \infty, \quad \lambda\theta \to 0, \quad \alpha/\theta \to 0, \quad \epsilon^{-3}\theta^{-1} \to 0 \tag{29}$$

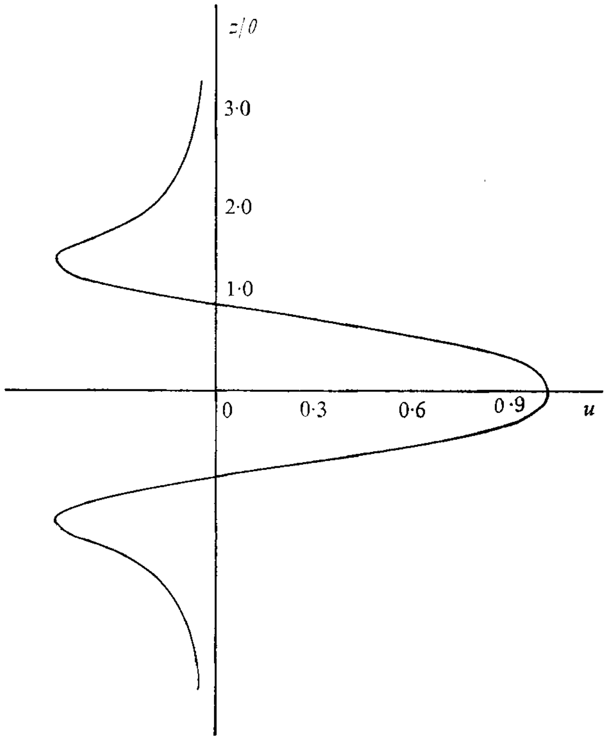

FIGURE 7. Horizontal velocity profile at distance from body corresponding to $\delta = (3x)^{-\frac{1}{3}}\,\theta = 3$.
There is actually a small excess velocity at the centre of the blocking column.

or in terms of the dimensional scales (§ 2)

$$\left.\begin{array}{c} b/l \to \infty, \quad b/h \to 0, \\ \dfrac{Ub}{\nu}\left(\dfrac{l}{b}\right)^2 \to 0, \quad \dfrac{U}{(gb)^{\frac{1}{2}}}\left(\dfrac{h}{b}\right)^{\frac{1}{2}}\left(\dfrac{\nu}{\vartheta}\right)^{\frac{1}{2}} \to 0. \end{array}\right\}$$

Thus the body thickness $2b$ must be large compared with the intrinsic scale l (table 2), but small compared with the stratification scale h. The 'Reynolds number' $\alpha = Ul/\nu$ can be large or even small, provided it is not too small. On the other hand, the body Froude number $F = U/(gb)^{\frac{1}{2}}$ must be quite small, especially when the Prandtl or Schmidt number $\sigma = \nu/\vartheta$ is large. In fact, the limit is essentially one in which the Boussinesq number $\beta = h/b$ and Richardson number $Ri = \beta^{-1}F^{-2} = b^2g/(U^2h)$ are large, since (29) can also be written

$$Re^{\frac{1}{3}}(\sigma Ri)^{\frac{1}{3}} \to \infty, \quad \beta \to \infty, \quad (\sigma Ri)^{-\frac{1}{2}} \to 0, \quad \sigma^{\frac{1}{2}} Ri^{-\frac{1}{2}} \to 0. \tag{30}$$

Here $Re = Ub/\nu$ can also be large (or even small, if not too small) so that (30) has the misleading appearance of an inviscid limit but, in fact, viscous shear has been seen to play a crucial role and inertia an unimportant one. By contrast, as $\beta \to \infty$ with $F^{-2} = \beta Ri$ fixed, (7)–(10) tend to the equations describing a homogeneous fluid.

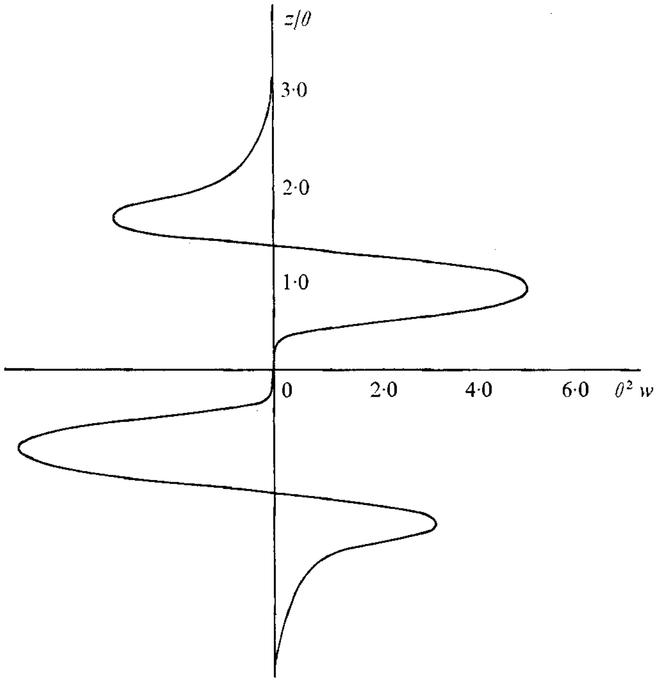

FIGURE 8. Vertical velocity profile at distance from body corresponding to $\delta = 3$. Note that the velocity scale is multiplied by θ^2, so that even at this distance, w itself is quite small.

For Yih's (1959) experiment, $\theta \approx 60$ and (29), (30) are well approximated (table 2), except for the last limit, since $\epsilon^{-3}\theta^{-1}$ ranged from $\frac{1}{2}$ to 5, so that the convection terms in the salinity diffusion balance were perhaps more important than allowed for in the present analysis.

The near-similarity approach has thus led to a likely approximation (27) which turns out to extend Long's (1962) far wake description nearly to the body surface. Indeed, we now notice that (29) and $\delta = (3x)^{-\frac{1}{3}}\theta \rightarrow 0$ together imply $x \rightarrow \infty$, $\alpha x^{-\frac{1}{3}} \rightarrow 0$, $\epsilon^{-3}x^{-\frac{1}{3}} \rightarrow 0$ and hence, imply (14), if η is fixed and λ, as usually, is very small indeed. The far wake is therefore effectively covered by the parameter limit (30), which need not be restricted by the qualification 'for fixed co-ordinate values' usually implicit in parameter limits.

It may be verified in a similar manner that Graebel's (1969) and Janowitz' (1971) non-diffusive proposals are limit solutions of the exact equations (7)–(10) in the parameter limit

$$\epsilon\theta \rightarrow \infty, \quad \lambda\theta \rightarrow 0, \quad (\alpha/\epsilon)(\epsilon\theta)^{-2} \rightarrow 0, \quad \epsilon^6\lambda\theta^3 \rightarrow 0$$

or $$(Re\,Ri)^{\frac{1}{3}} \rightarrow \infty, \quad \beta \rightarrow \infty, \quad Ri \rightarrow \infty, \quad \sigma\beta\,Ri^{-1} \rightarrow \infty, \tag{31}$$

which differs from (29) and (30) mainly in the fourth condition. Thus (31) puts an upper bound on the Richardson number, while (30) does not, and since $Ri \propto b^2$, (30) must become more appropriate (all other things being equal) for sufficiently thick bodies. Of course, even though they are limit solutions, the proposals of

Graebel (1969) and Janowitz (1971) fail to predict a downstream wake† (Yih 1959) and a finite drag dependent on the body speed.

6. Structure of Taylor columns

The two-dimensional motion of a viscous, incompressible, homogeneous fluid in a system rotating with constant angular velocity Ω is known (Veronis 1970) to be related by an analogy to the two-dimensional motion of a viscous, diffusive, incompressible and stratified fluid of unit Schmidt or Prandtl number to which the Boussinesq approximation has been applied absolutely ($\lambda = 0$). The motion in the rotating system is here understood to be two-dimensional in the sense that it does not depend on the distance y' from a plane through the axis of rotation. If the angular velocity is in the direction of increasing z', the analogy is expressed by the correspondence

$$\begin{array}{cccccccccc} x & z & u & w & \rho & p & l & \alpha = \epsilon^{-3} & U \\ z' & x' & w' & u' & -v' & p' & l' & \alpha' & V \end{array} \tag{32}$$

where primes distinguish the quantities referring to the rotating system, $l' = (\frac{1}{2}\nu/\Omega)^{\frac{1}{2}}$ and $\alpha' = V/(2\nu\Omega)^{\frac{1}{2}}$; a transformation to non-dimensional variables precisely analogous to (6) (except that $\rho' = \text{const} = \rho_0$) has been applied to the variables in the rotating system. In other words, the motion in the rotating system is governed by (7)–(10) translated according to (32), except that $\lambda = 0$, $\epsilon^{-3} = \alpha'$ and $\tilde{\rho} = 1$. The analogue of the boundary condition (2) is that the fluid motion approaches rest (with respect to the rotating system) as $|z'| \to \infty$, and the horizontal motion of a vertical plate of height $2b$ in stratified fluid corresponds to the broadside motion with constant speed V along the axis of rotation of a plate of width $2b' = 2\theta' l'$ in the x' direction and of indefinite extent in the y' direction.

It follows that the analogue of (27) is a near-similar limit solution of the *exact* equations governing viscous, homogeneous fluid motion in the rotating system in the limit

$$\left.\begin{aligned} \theta' &= b'/l' = b'(2\Omega/\nu)^{\frac{1}{2}} = (2/E)^{\frac{1}{2}} \to \infty, \\ \alpha'/\theta' &= V/(2\Omega b') = \tfrac{1}{2}Ro \to 0, \end{aligned}\right\} \tag{33}$$

where Ro and $E = \nu/(\Omega b'^2) = Ro/Re$ are the Rossby and Ekman numbers, respectively. This limit solution describes the Taylor columns fore and aft of the plate as long wakes. They approach similarity form analogous to (12) as $\delta' = (3z')^{-\frac{1}{3}}\theta' = [2\Omega/(3\nu z^*)]^{\frac{1}{3}} b' \to 0$ and then consist of a weak central jet in the direction of the plate's motion, flanked by pairs of still weaker jets of alternating direction – the lateral decay of the column is oscillatory and algebraic (§ 3). The axial decay is described by (12) and (32). But since $\delta' \to 0$ only for $|z'| \gg \theta'^3$, a large distance may be needed for a close quantitative approach to this similarity form.

The limit solution satisfies the condition of zero normal velocity on the plate, but fails to be a limit solution of the governing equations close to the plate and,

† See Note in review on page 715.

especially, in a neighbourhood of the plate edges. However, it describes the transition from the similarity decay law to a well-formed Taylor column (Taylor 1922) not far from the plate. The 'edges' of the column are distinguished by strong velocity gradients, with velocities parallel to the plane of the plate predominant in the inner part of the 'layer' and axial velocities, in the outer part. These bounding 'layers', and the column with them, spread gradually with distance from the plate. The ultimate spread is the analogue of that described by (12), (13) and (15), (16) with $\eta' = (3z')^{-\frac{1}{3}}x' = [2\Omega/(3\nu z^*)]^{\frac{1}{3}}x^*$ in the place of η.

The mass-flow in the column is balanced by a back-flow in the bounding layer. The analogy gives the drag per unit span of the plate as $\pi\mu V(b'/l')^2$, i.e. gives a drag coefficient $D/(\frac{1}{2}\rho_0 V^2 \times \text{area}) = 2\pi/Ro$. This is about 21 % higher than observed for a disk (Maxworthy 1970), the difference being due to the important differences between two-dimensional and axisymmetrical motion (Moore & Saffman 1969, whose analysis shows the drag coefficient then to be $32\Omega r/(3\pi V)$ for a disk of radius r) at distances $O(\Omega b'^3/\nu)$ from the body, where the Taylor column disappears. The predictions are also consistent with those of Bretherton (1967) for the temporal development of two-dimensional Taylor columns in unbounded fluid.

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Appendix A

To compute the momentum integral of Long's (1962) similarity solution from (16), note that the Airy function is defined by

$$Ai\,(z) = \frac{1}{2\pi i}\left\{\int_{P_1} - \int_{P_2}\right\}\exp\,(zt - \tfrac{1}{3}t^3)\,dt,$$

where P_k are the rays from 0 to $\infty \exp \frac{2}{3}\pi i k$, and similarly (Scorer 1950)

$$Gi(z) = \frac{-1}{2\pi}\left\{\int_{P_1} + \int_{P_2}\right\}\exp\,(zt - \tfrac{1}{3}t^3)\,dt.$$

Thus

$$\begin{aligned}\frac{h(\eta)}{h''(0)} &= \frac{i\pi}{2}\left[\frac{-j}{\pi}\int_{P_2}\exp\,(\xi t - \tfrac{1}{3}t^3)\,dt + \frac{1}{\pi j}\int_{P_1}\exp\,(\tau t - \tfrac{1}{3}t^3)\,dt\right] \\ &= -\frac{1}{2}\int_0^\infty \exp\,(-is\eta - \tfrac{1}{3}s^3)\,ds - \frac{1}{2}\int_0^\infty \exp\,(is\eta - \tfrac{1}{3}s^3)\,ds \\ &= -\int_0^\infty \exp\,(-\tfrac{1}{3}s^3)\cos\,(\eta s)\,ds\end{aligned}$$

and

$$\int_{-\infty}^\infty h(\eta)\,d\eta = -2h''(0)\int_0^\infty d\eta\int_0^\infty \exp\,(-\tfrac{1}{3}s^3)\cos\,(\eta s)\,ds,$$

so that (18) will follow from the more general result

$$I_n \equiv \int_0^\infty d\eta \int_0^\infty \exp(-t^n/n)\cos(\eta t)\,dt = \tfrac{1}{2}\pi \quad \text{for} \quad n > 0.$$

Here integration by parts gives

$$I_n = \lim_{a\to\infty} \int_0^a g(\eta)\,d\eta, \quad g(\eta) = \int_0^\infty t^{n-1}\exp(-t^n/n)\frac{\sin(\eta t)}{\eta}\,dt$$

and $g(\eta)\in L(0,a)$ for any fixed $a > 0$, since it is defined and bounded on $(0,a)$. By Fubini's theorem, therefore

$$I_n = \lim_{a\to\infty}\int_1^\infty t^{n-1}\exp(-t^n/n)\,Si(at)\,dt,$$

where
$$Si(z) = \int_0^z x^{-1}\sin x\,dx$$

is bounded on $[0,\infty)$. Thus for $a > 0$,

$$|t^{n-1}\exp(-t^n/n)\,Si(at)| \leqslant t^{n-1}\exp(-t^n/n)\times\text{constant}\in L(0,\infty)$$

and by Lebesgue's dominated convergence theorem,

$$I_n = \int_0^\infty t^{n-1}\exp(-t^n/n)\lim_{a\to\infty} Si(at)\,dt = Si(\infty) = \pi/2.$$

The same result permits the exact computation of the momentum integral of Long's (1959) non-diffusive far wake as

$$2^3\mu U I_4 = 4\pi\mu U.$$

For even $n > 0$, I_n was effectively computed by Bernstein (1919) by a different approach. The present method also yields

$$I_{k,n} = \int_0^\infty dx \int_0^\infty t^k \exp(-t^n/n)\cos(xt)\,dt = 0$$

for $k \geqslant 1$, $n \geqslant 1$.

Appendix B

To obtain the limit of (27) as $\delta = (3x)^{-\frac{1}{3}}\theta \to +\infty$, observe first that

$$s^{-1}J_1(s)\cos(sz/\theta)\in L(0,\infty)$$

for any z/θ, so that (28) gives, by the dominated convergence theorem,

$$\lim_{\delta\to\infty} p(x,z) = \theta\int_0^\infty s^{-1}J_1(s)\cos(sz/\theta)\,ds$$
$$= \begin{cases} (\theta^2 - z^2)^{\frac{1}{2}} & \text{for} \quad |z| < \theta \\ 0 & \text{for} \quad |z| > \theta \end{cases}$$

(Watson 1966, p. 405). Next, with $\Delta = 1/\delta$, (28) gives

$$u(x,z) = \int_0^\infty \phi(s,\Delta)\,\chi(s)\,ds, \quad \phi(s,\Delta) = s^{-\frac{1}{2}}\exp(-s^3\Delta^3/3)$$
$$\chi(s) = s^{\frac{1}{2}}J_1(s)\cos(sz/\theta).$$

and $\phi(s,\Delta)$ is positive and monotone decreasing to zero as s increases for every Δ in $[0,1]$, while $\chi(s)$ is continuous and integrable on $(0,\infty)$ for $z/\theta \neq \pm 1$. By Dirichlet's test for uniform convergence (Apostol 1957), therefore, the integral for u converges uniformly with respect to Δ in $[0,1]$, except when $z = \pm\theta$, so that

$$\lim_{\delta\to\infty} u = \int_0^\infty \phi(s,0)\,\chi(s)\,ds = \int_0^\infty J_1(t)\cos(tz/\theta)\,dt \quad (|z| \neq \theta)$$
$$= \begin{cases} 1 & \text{for} \quad |z| < \theta \\ -\theta^2(z^2-\theta^2)^{-\frac{1}{2}}[|z| + (z^2-\theta^2)^{\frac{1}{2}}]^{-1} & \text{for} \quad |z| > \theta \end{cases}$$

(Watson 1966). The same argument can be applied to $\rho(x,z)$ in (28), but either approach fails for $w(x,z)$. However, in the space D' of generalized functions on the test functions of compact support, we may compute from (11)

$$\lim_{\delta\to\infty} w = (\partial^2/\partial z^2)\lim_{\delta\to\infty}\rho.$$

Appendix C. Non-diffusive theory

Neglect of diffusion ($\vartheta/\nu = 0$) requires a rescaling to (§ 2)

$$d = (hU\nu/g)^{\frac{1}{3}} = l/\epsilon,$$
$$\bar{x} = x^*/d = \epsilon x, \quad \bar{z} = z^*/d = \epsilon z,$$
$$u^* = -U + U\bar{u}(\bar{x},\bar{z}), \quad w^* = U\bar{w}(\bar{x},\bar{z}),$$
$$\rho^* = \rho_0\breve{\rho} = \rho_\infty + \epsilon^{-1}\lambda\rho_0\bar{\rho}(\bar{x},\bar{z}),$$
$$p^* = p_{\mathrm{st}} + (\mu U/d)\,\bar{p}(\bar{x},\bar{z}).$$

From (5), the density is seen to be convected, and from (3) the motion is therefore seen to have a stream function

$$\psi^* = -Uz^* + Ud\bar{\psi}(\bar{x},\bar{z})$$
$$\bar{u} = \partial\bar{\psi}/\partial\bar{z}, \quad \bar{w} = -\partial\bar{\psi}/\partial\bar{x}. \tag{34}$$

Thus $\rho^* = \rho^*(\psi^*)$, which is determined by (1) and (2) as

$$\rho^* = \rho_0(1+\psi^*/(Uh)) = \rho_\infty + \rho_0\epsilon^{-1}\lambda\bar{\psi},$$
$$\breve{\rho} = 1 - \epsilon^{-1}\lambda(\bar{z}-\bar{\psi}),$$

and (4) gives

$$(\alpha/\epsilon)\,\bar{D}\bar{u} = -\partial\bar{p}/\partial\bar{x} + \bar{\nabla}^2\bar{u}, \tag{35}$$
$$(\alpha/\epsilon)\,\bar{D}\bar{w} = -\partial\bar{p}/\partial\bar{z} + \bar{\nabla}^2\bar{w} - \bar{\psi}, \tag{36}$$

where $\alpha/\epsilon = Ud/\nu$, $\bar{D} = \breve{\rho}(\bar{u}-1)\,\partial/\partial\bar{x} + \breve{\rho}\bar{w}\partial/\partial\bar{z}$ and $\bar{\nabla}^2 = \partial^2/\partial\bar{x}^2 + \partial^2/\partial\bar{z}^2$.

It may be seen from the following three arguments that (34)–(36) are unlikely to admit solutions with fore-and-aft symmetry. First, $\bar{u}(-\bar{x},\bar{z}) = \bar{u}(\bar{x},\bar{z})$ implies, with the choice $\psi^*(x^*,0) = 0$, that $\bar{\psi}$ is even in $\bar{x}$ and $\bar{w}$, odd. Thus $\bar{\nabla}^2\bar{w}$ and

$$(\bar{u}-1)\,\partial\bar{u}/\partial\bar{x}+\bar{w}\,\partial\bar{u}/\partial\bar{z} = \tilde{\rho}^{-1}\bar{D}\bar{u}$$

are odd, while $\bar{\nabla}^2\bar{u}$ and $\rho^{-1}\bar{D}\bar{w}$ are even. Since $\tilde{\rho} = 1-\epsilon^{-1}\lambda(\bar{z}-\bar{\psi})$ is even, it follows that $\bar{D}\bar{u}$ is odd and $\bar{D}\bar{w}$, even. Now, if $\bar{p}(\bar{x},\bar{z})$ be split into its even part $\bar{p}_e$ and odd part $\bar{p}_d$, then the even part of (35) and odd part of (36) are

$$\partial\bar{p}_d/\partial\bar{x} = \bar{\nabla}^2\bar{u}, \quad \partial\bar{p}_d/\partial\bar{z} = \bar{\nabla}^2\bar{w} \tag{37}$$

while the odd part of (35) and even part of (36) are

$$(\alpha/\epsilon)\,\bar{D}\bar{u} = -\,\partial\bar{p}_e/\partial\bar{x}, \quad (\alpha/\epsilon)\,\bar{D}\bar{w} = -\,\partial\bar{p}_e/\partial\bar{z}-\bar{\psi}. \tag{38}$$

On transforming back to the original variables, (37) is seen to state that ψ^* describes the slow, viscous Stokes flow past the body in homogeneous fluid. By contrast, (38) is seen to state that the same ψ^* describes inviscid motion past the body in the stratified fluid.

Secondly, we may inquire whether solutions can exist which, though not symmetrical near the body, approach fore-and-aft symmetry in the wakes far from the body. It is then plausible (Long 1959; Bretherton 1967) to neglect inertia in the far wakes, and the resulting comparison equations corresponding to (11) are (Long 1959; Graebel 1969; Janowitz 1971) readily found from (34) to (36) to be

$$\begin{aligned} \bar{u} &= \partial\bar{\psi}/\partial\bar{z}, \quad \bar{w} = -\,\partial\bar{\psi}/\partial\bar{x}, \\ \partial\bar{p}/\partial\bar{x} &= \partial^2\bar{u}/\partial\bar{z}^2, \quad \partial\bar{p}/\partial\bar{z} = -\bar{\psi}, \end{aligned} \tag{39}$$

which cannot have a solution with $\bar{u}$ even in $\bar{x}$ (Graebel 1969).

Thirdly, if inertia is not neglected entirely, the Oseen approximation to (34)–(36), in which $\bar{D}$ is replaced by $-\,\partial/\partial\bar{x}$, also leads to the same conclusion. Indeed, (38) then becomes

$$\bar{p}_e-\alpha\bar{u}/\epsilon = g(\bar{z}), \quad g'(\bar{z})+\bar{\psi} = -(\alpha/\epsilon)\,\bar{\nabla}^2\bar{\psi},$$

whence $g(\bar{z})$ = constant because $\bar{\psi}\to 0$ and $\bar{\nabla}^2\bar{\psi}\to 0$ as $|x|\to\infty$. But that implies $\bar{\psi}\equiv 0$ because (37) implies $\bar{\nabla}^4\bar{\psi} = 0$. Hence, there is no such Oseen wake leaving the fluid undisturbed at infinity.

Janowitz' (1971) proposal for an approximate solution of (31) to (33) representing steady, two-dimensional flow past a vertical plate is

$$\begin{aligned} \bar{\psi}(\bar{x},\bar{z}) &= \bar{\theta}\int_0^\infty t^{-1}J_1(t)\exp\left[-\bar{x}(t/\bar{\theta})^4\right]\sin\left(t\bar{z}/\bar{\theta}\right)dt \\ &= \bar{\theta}\int_0^\infty r^{-1}J_1(\Delta r)\exp\left[-\tfrac{1}{4}r^4\right]\sin\left(\bar{\eta}r\right)dr \end{aligned}$$

for $\bar{x} > 0$, $-\infty < \bar{z} < \infty$, where

$$\bar{\theta} = b/d, \quad \Delta = (4\bar{x})^{-\frac{1}{4}}\bar{\theta}, \quad \bar{\eta} = (4\bar{x})^{-\frac{1}{4}}\bar{z}.$$

This is a near-similar solution of (39) for $\bar{x} > 0$. Indeed, $J_1(y) = \frac{1}{2}y+O(y^3)$ for small $|y|$, and since $|J_1(y)| \leqslant 1$ in any case, and $1+\frac{1}{2}\Delta r < r^4$ for $\Delta < \frac{1}{2}$ and $r \geqslant 2^{\frac{1}{2}}$,

$$
\begin{aligned}
\Delta^{-3}\left|\overline{\psi} - \tfrac{1}{2}\overline{\theta}\Delta\int_0^\infty \exp\left(-\tfrac{1}{4}r^4\right)\sin\left(\overline{\eta}r\right)dr\right|
&\leqslant \Delta^{-3}\overline{\theta}\left(\int_0^{\Delta^{-\frac{1}{2}}} + \int_{\Delta^{-\frac{1}{2}}}^\infty\right)\exp\left(-\tfrac{1}{4}r^4\right)\left|J_1(\Delta r) - \tfrac{1}{2}\Delta r\right|\frac{dr}{r} \\
&< C\overline{\theta}\int_0^\infty r^2\exp\left(-\tfrac{1}{4}r^4\right)dr + \Delta^{-3}\overline{\theta}\int_{\Delta^{-\frac{1}{2}}}^\infty r^3\exp\left(-\tfrac{1}{4}r^4\right)dr \\
&= C_1\overline{\theta} + \Delta^{-3}\overline{\theta}\exp\left(-(2\Delta)^{-2}\right)
\end{aligned}
$$

for sufficiently small Δ, so that

$$
\overline{\psi}(\overline{x},\overline{z}) = \tfrac{1}{2}\overline{\theta}\Delta\int_0^\infty \exp\left(-\tfrac{1}{4}r^4\right)\sin\left(\overline{\eta}r\right)dr + O(\Delta^3)
$$

as $\Delta \to 0$ in the far upstream 'wake'. By (39), therefore

$$
\begin{aligned}
\overline{p}(\overline{x},\overline{z}) &= -\int^{\overline{z}} \overline{\psi}(\overline{x},s)\,ds \\
&= \tfrac{1}{2}\overline{\theta}^2\int_0^\infty r^{-1}\exp\left(-\tfrac{1}{4}r^4\right)\cos\left(\overline{\eta}r\right)dr + O(\overline{x}^{-\frac{1}{2}}) + \text{constant},
\end{aligned}
$$

which does not exist as it stands, but can be regularized to

$$
\overline{p}(\overline{x},\overline{z}) = \tfrac{1}{2}\overline{\theta}^2\int_0^\infty \left[r^{-1}\exp\left(-\tfrac{1}{4}r^4\right)\cos\left(\overline{\eta}r\right) - \alpha(r)\right]dr + O(\overline{x}^{-\frac{1}{2}})
$$

with $\alpha(r)$ independent of $\overline{\eta}$ and such that $\int_0^\infty \alpha(r)\,dr$ exists for any $\epsilon > 0$ and $\alpha(r) - r^{-1}$ is bounded as $r \to 0$. Even then, however,

$$
\begin{aligned}
\overline{p}(\overline{x},\overline{z}) - \overline{p}(\overline{x},0) &= -\overline{\theta}^2\int_0^\infty r^{-1}\exp\left(-\tfrac{1}{4}r^4\right)\sin^2\left(\tfrac{1}{2}\overline{\eta}r\right)dr \\
&\to -\infty \quad \text{as} \quad |\overline{\eta}| \to \infty,
\end{aligned}
$$

because

$$
\int_0^\epsilon t^{-1}\sin^2(kt)\,dt \to \infty \quad \text{as} \quad k \to \infty \quad \text{for any} \quad \epsilon > 0.
$$

Hence

$$
\int_{-\infty}^\infty \overline{p}(\overline{x},\overline{z})\,d\overline{z}
$$

cannot exist, and as with diffusive similarity solutions, the other terms in the momentum integrand (§ 3) decay faster with distance from the body than the pressure perturbation. Thus the momentum integral across the upstream wake diverges. Moreover, there is no downstream wake to balance the divergence and hence a finite drag per unit span of body cannot be expected.

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